*International Journal of Theoretical Physics, Vol. 36, No. 6, 1997*

# Dissipative Fluid in Conformally Flat Space-Time

**M. A. P. Martins,[1] N. O. Santos,[2] and M. M. Som[1]**

*Received February 18, 1997*

We consider a conformally flat, inhomogeneous solution of the Einstein equations for a dissipative fluid. The production of entropy is found to depend on some arbitrary functions of time. By some subsidiary conditions, such a model is shown to evolve into a homogeneous Friedmann-type universe.

## 1. INTRODUCTION

The perfect-fluid cosmology with spatially homogeneous isotropic universe governed by Einstein's equations represents fairly well the large-scale structure of the present-day universe. Looking back in time, one finds that such a universe must have a singular origin of infinite density of matter. Apart from the singular origin, one encounters a series of problems such as the particle horizon, the formation of galaxies, and a high photon–baryon ratio. To circumvent these problems in the classical context one looks for some physical process that can account for these problems reasonably well. It is natural to consider then a deviation from the perfect-fluid content of the universe. At the very early universe, one might expect that pressure, density, or velocity would vary over distances of the order of a mean free path or over times of the order of a mean free time or both (Weinberg, 1971). In such a case dissipative effects might play an important role in the evolution of the universe.

Perfect fluids with bulk viscosity have been studied (e.g., Weinberg, 1971; Murphy, 1973; Nightingale, 1973; and references therein). In a short communication, Som and Santos (1980) investigated the role of dissipation

[1] Instituto de Física, Universidade Federal do Rio de Janeiro, C.P. 68528, Ilha do Fundão, Rio de Janeiro, 21945-970, RJ, Brazil.
[2] Grupo de Astrofísca Relativística, Observatório Nacional, CNPq, 586-20921, Rio de Janeiro, Brazil.

of the fluid kinetic energy as heat flow. A spatially homogeneous space-time does not admit heat flow. The spatially homogeneous Robertson–Walker space-time which is conformally flat allows dissipative processes related to bulk viscosity. However, the bulk viscosity contributes negligibly to the production of entropy when the pressure and thermal energy are dominated by radiation. If one maintains conformal flatness of space-time, the only alternative remaining to study the effect of other dissipative processes is the inhomogeneous, conformally flat space-time. Any conformally flat, inhomogeneous space-time where the entropy production might take place through heat flow might be a good candidate if it tends asymptotically to the Robertson–Walker space-time.

In this paper, we investigate the role of dissipative processes in a new conformally flat, inhomogeneous space-time. As the shear and vorticity vanish identically in such a space-time, dissipative processes are mediated by means of the bulk viscosity and heat flow only.

## 2. THE GENERAL EQUATIONS OF MOTION

The energy momentum tensor of a fluid with bulk viscosity and heat flow is given by (Greek and Roman indices run from 0 to 3 and from 1 to 3, respectively)

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + (p + \tau) h^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} \tag{2.1}$$

In this expression $u^{\mu}$ is the four-velocity of the particles in a fluid of energy density $\rho$ and pressure $p$. The tensor $h^{\mu\nu}$ is a projection tensor defined as

$$h^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu} \tag{2.2}$$

The fields $\tau$ and $q^{\mu}$ describe the out-of-equilibrium properties of the fluid. The field $q^{\alpha}$ satisfies the constraint

$$u^{\alpha} q_{\alpha} = 0 \tag{2.3}$$

The conservation of energy-momentum in such a field is given by

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2.4}$$

where the semicolon denotes covariant derivative.

Treating the number density of particles in the fluid $n$ as a fundamental thermodynamic variable along with $\rho$, we have for the conservation of particle current $N^{\mu}$

$$N^{\mu}{}_{;\mu} = (n u^{\mu})_{;\mu} = 0 \tag{2.5}$$

The entropy per particle $s$ is specified by the equation of state

$$s = s(\rho, n) \tag{2.6}$$

The temperature and pressure are obtained from the second law of thermodynamics,

$$T\,ds = \frac{1}{n}\left[d\rho - \frac{\rho + p}{n}\,dn\right] \tag{2.7}$$

$$\frac{1}{T} = n\left(\frac{\partial s}{\partial \rho}\right)_n \tag{2.8}$$

$$p = -\rho - n^2 T\left(\frac{\partial s}{\partial n}\right)_\rho \tag{2.9}$$

Since the system is out of equilibrium, one must have $S^\alpha{}_{;\alpha} \geq 0$ from the second law of the thermodynamics, where $S^\alpha$ is the entropy current.

The standard theory of Eckart (1940) uses the following expression for the entropy current

$$S^\alpha = snu^\alpha + \frac{q^\alpha}{T} \tag{2.10}$$

Such a formulation is inadequate for many phenomena involving steep space-time gradients of heat flux and viscous stress. However, under quasi-stationary conditions, i.e., when these fields vary slowly on space-time scales characterized by the mean free path and mean free time, the formulation of Eckart is correct up to the first order in deviations from equilibrium.

Under quasistationary conditions, one obtains, using (2.4) and (2.5),

$$TS^\alpha{}_{;\alpha} = -[\tau u^\alpha{}_{;\alpha} + q^\alpha(T^{-1}T_{,\alpha} + u^\beta u_{\alpha;\beta})] \tag{2.11}$$

where the colon denotes ordinary partial derivative.

The second law of thermodynamics $S^\alpha_{;\alpha} \geq 0$ then requires that

$$\tau = -\zeta u^\alpha{}_{;\alpha} = -\zeta\theta \tag{2.12}$$

$$q^\alpha = -\chi h^{\alpha\beta}(T_{,\beta} + Tu^\nu u_{\beta;\nu}) \tag{2.13}$$

where $\theta = u^\alpha{}_{;\alpha}$ is the expansion scalar, and $\zeta > 0$ and $\chi > 0$ are the phenomenological coefficients of the bulk viscosity and the thermal conductivity respectively, characteristics of the material medium. With these expressions one then obtains the divergence of the entropy current as positive definite,

$$S^\alpha{}_{;\alpha} = \frac{1}{T}\left[\zeta\theta^2 + \left(\frac{\chi}{T}\right)q^\alpha q_\alpha\right] \tag{2.14}$$

Equations (2.12) and (2.13) along with (2.4) and (2.5) form a complete system of equations for the dynamical variables ($\rho$, $n$, $\tau$, $u^\alpha$, and $q^\alpha$).

## 3. A NEW CONFORMALLY FLAT SOLUTION OF THE EINSTEIN EQUATIONS FOR A FLUID WITH BULK VISCOSITY AND HEAT FLOW

In this section we shall obtain a new exact solution of Einstein's equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{3.1}$$

in the form

$$g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \qquad \sigma = \sigma(x^\alpha) \tag{3.2}$$

where $T_{\mu\nu}$ corresponds to (2.1).

With

$$u^\mu = e^{-\sigma}\delta_0^\mu \tag{3.3}$$

one gets from (2.12) and (3.2)

$$\theta = -3[e^{-\sigma}]_{,0} \tag{3.4}$$

The Einstein equations (3.1) can be written down explicitly (Eisnenhart, 1926) with the help of (2.1) and (3.2)–(3.4). The result is

$$G_{00} = \frac{2}{3}\theta_{,0}e^{\sigma} + \Delta_1\sigma + 2\Delta_2\sigma = \kappa\rho e^{2\sigma} \tag{3.5}$$

$$G_{0j} = \frac{2}{3}\theta_{,j}e^{\sigma} = -\kappa q_j e^{\sigma} \tag{3.6}$$

$$G_{ij} = 2\sigma_{ij} - \delta_{ij}(\Delta_1\sigma + 2\Delta_2\sigma) = \kappa e^{2\sigma}\delta_{ij}(p + \tau) \tag{3.7}$$

where

$$\sigma_{\mu\nu} = \sigma_{,\mu\nu} - \sigma_{,\mu}\sigma_{,\nu} \tag{3.8}$$

$$\Delta_1\sigma = \eta^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} \tag{3.9}$$

$$\Delta_2\sigma = \eta^{\mu\nu}\sigma_{,\mu\nu} \tag{3.10}$$

Adding up equations (3.5) and (3.7) for $i = j$, we have after some rearrangements

$$(e^{-\sigma})_{,11} = \frac{1}{3}\theta_{,0} - \frac{\kappa}{2}(e^{-\sigma})(p + \tau + \rho) \tag{3.11}$$

This set of equations allow us to write

$$(e^{-\sigma})_{,11} = (e^{-\sigma})_{,22} = (e^{-\sigma})_{,33} \tag{3.12}$$

Thus $(e^{-\sigma})$ must assume the general form

$$e^{-\sigma} = \frac{1}{2} \lambda \delta_{ij} x^i x^j + \alpha_i x^i + \beta \tag{3.13}$$

where $\lambda$, $\alpha$, and $\beta$ are arbitrary functions of $x^0 = t$.

From (3.7), for $i \neq j$, we get

$$(e^{-\sigma})_{,ij} = 0 \tag{3.14}$$

which is consistent with (3.13).

The expansion scalar is given by

$$\theta = -3[\tfrac{1}{2}\dot{\lambda}\delta_{ij} x x^j + \dot{\alpha}_j x^j + \dot{\beta}] \tag{3.15}$$

where the dot represents derivative with respect to $x^0$.

The heat flow $q_\mu$ relative to $u^\mu$ can be obtained from (3.6) and (3.15), resulting in

$$\kappa q_i = \delta_{ij}(\dot{\lambda} x^j + \dot{\alpha}^j) \tag{3.16}$$

The total energy density of matter measured by $u^\mu$ is

$$\kappa \rho = 3[(\tfrac{1}{2}\dot{\lambda}\delta_{ij} x^i x^j + \dot{\alpha}_i x^i + \dot{\beta})^2 - \delta_{ij}\delta_{ik}(\lambda x^j + \alpha^j)(\lambda x^k + \alpha^k) + 2\lambda e^{-\alpha}] \tag{3.17}$$

Finally, the effective pressure $\tilde{p} = (p + \tau)$ due to the presence of bulk viscosity is given by

$$\kappa \tilde{p} = 2(\tfrac{1}{2}\ddot{\lambda}\delta_{ij} x^i x^j + \ddot{\alpha}_j x^j + \ddot{\beta}) - 3[(\tfrac{1}{2}\dot{\lambda}\delta_{ij} x^i x^j + \dot{\alpha}_j x^j + \dot{\beta})^2 - \delta_{ij}\delta_{ik}(\lambda x^j + \alpha^j)(\lambda x^k + \alpha^k) - 4\lambda e^{-\sigma}] \tag{3.18}$$

## 4. ENTROPY PRODUCTION

From equations (2.7) and (2.8) one obtains, since $ds$ is a perfect differential,

$$T\left(\frac{\partial p}{\partial T}\right)_n = p + \rho - n\left(\frac{\partial \rho}{\partial n}\right)_T \tag{4.1}$$

We now consider a material medium of short mean free path and mean free time interacting with radiation with mean free time $t_m$, in the quasistationary state. Then for adiabatic motions, using (2.5), one obtains from (2.7)

$$u^\alpha\left(\frac{\partial T}{\partial x^\alpha}\right) = \left(\frac{\partial \rho}{\partial T}\right)_n^{-1}\left[n\left(\frac{\partial \rho}{\partial n}\right)_T - \rho - p\right] \tag{4.2}$$

Substituting (4.1) in (4.2), one gets

$$u^{\alpha}\left(\frac{\partial T}{\partial x^{\alpha}}\right) = -\left(\frac{\partial \rho}{\partial T}\right)_n^{-1} T\left(\frac{\partial p}{\partial T}\right)_n u^{\beta}_{;\beta} + O(t_m) \tag{4.3}$$

The coefficients $\zeta$ and $\chi$ take the form

$$\zeta = 4aT^4 t_m\left[\frac{1}{3} - \left(\frac{\partial p}{\partial \rho}\right)_n\right]^2 \tag{4.4}$$

$$\chi = \frac{4}{3} aT^3 t_m \tag{4.5}$$

where $a$ is the Stefan–Boltzmann constant.

From (4.4) and (4.5) one obtains

$$\zeta = 3\chi T\left[\frac{1}{3} - \left(\frac{\partial p}{\partial \rho}\right)_n\right]^2 \tag{4.6}$$

With the equations (4.4) and (4.5) we have now a well-determined system with suitable initial conditions.

## 5. CONCLUSION

The positivity of the total energy density of matter gives a relation of constraint among the arbitrary functions $\lambda$, $\alpha$, and $\beta$ of time, such that $2\lambda\beta - \delta_{ij}\alpha^i\alpha^j \geq 0$. If $\lambda = \alpha_i = 0$, one obtains the usual homogeneous and isotropic solutions of a fluid with bulk viscosity. A particular case of the homogeneous and isotropic solution corresponds to the de Sitter phase when $\dot{\beta} = \text{const}$. This result has been obtained by Som and Berman (1989).

It is evident from (4.6) that the entropy production depends on the characteristics of viscosity and heat conduction. If the interacting medium is highly relativistic, then $(\partial p/\partial \rho) \to 1/3$ and $\zeta \to 0$. So, even in this case, the entropy might arise due to heat conduction. An interesting possibility arises if one introduces some subsidiary conditions to remove the arbitrariness of $\lambda$ and $\alpha_i$ such that $\lambda$ and $\alpha_i$ tend to zero for large values of time, consistent with the condition $2\beta\lambda - \delta_{ij}\alpha^i\alpha^j \geq 0$. One of the subsidiary conditions might be introduced in the form of the flux of heat vanishing for a large value of time. The universe, starting from a highly inhomogeneous and anisotropic phase, when the production of entropy is quite high due to heat flow and bulk viscosity, evolves into a homogeneous and isotropic Friedmann-type universe. However, note that we have considered only the first-order dissipative process. Unless the viscosity and the heat conduction vary slowly on space-time scales characterized by the mean free path and the mean free time, equations (2.12) and (2.13) violate causality. For a steep variation of these fields one must consider the second-order dissipative process in which the propagation equations for viscosity and heat conduction are hyperbolic (Anile and Choquet-Bruhat, 1989).

## ACKNOWLEDGMENTS

The authors wish to thank CNPq and FINEP for financial assistance to realize the present work.

## REFERENCES

Anile, A., and Choquet-Bruhat, Y., eds. (1989). *Relativistic Fluid Dynamics,* Springer, Berlin.

Eckart, C. (1940). *Physical Review,* **58**, 919.

Eisehart, L. P. (1926). *Riemannian Geometry,* Princeton University Press, Princeton, New Jersey, p. 90.

Murphy, G. L. (1973). *Physical Review D,* **8**, 4231.

Nightingale, J. D. (1973). *Astrophysical Journal,* **184**, 105.

Som, M. M., and Berman, M. S. (1989). *Progress of Theoretical Physics,* **81**, 823.

Som, M. M., and Santos, N. O. (1980). *Journal of Physics A,* **13**, 191.

Weinberg, S. (1971). *Astrophysical Journal,* **168**, 175.